# Effects of surface, wedge, corner, and mixed boundary conditions on the local critical behaviour



1990 J. Phys. A: Math. Gen. 23 2575

(http://iopscience.iop.org/0305-4470/23/12/033)

J. Phys. A: Math. Gen. 23 (1990) 2575-2585. Printed in the UK

# Effects of surface, wedge, corner, and mixed boundary conditions on the local critical behaviour

Zhen-Gang Wang†‡, Adolfo M Nemirovsky§, Karl F Freed|| and Kurt R Myers||

† Corporate Research Science Laboratories, Exxon Research and Engineering Company, Annandale, NJ 08801, USA
§ Departamento de Fisica, Universidade Federal de Pernambuco, 50739 Recife, PE, Brazil
|| The James Franck Institute, University of Chicago, Chicago, IL 60637, USA

Received 7 August 1989, in final form 15 March 1990

**Abstract.** Critical exponents for the O($n$) spin model local field near a surface, an edge and a corner are calculated by means of the renormalisation group method for the special geometry where the wedge or corner is formed by mutually perpendicular $(d-1)$-dimensional walls, each having von Neumann (special transition) or Dirichlet (ordinary transitions) boundary conditions. The combined effect of two or more interacting surfaces leads to new local susceptibility exponents which are calculated to order $\varepsilon$ $(\varepsilon = 4-d)$. The case of two or more von Neumann surfaces presents some difficulty in that a double divergence arises in the *first-order* perturbation calculation of the local susceptibilities.

## 1. Introduction

Geometric shapes and surface interactions play important roles in many physical systems. The most familiar examples of these systems can be found in the context of electrostatics (Jackson 1975) and heat conduction (Carslaw and Jaeger 1959). In recent years, these shape and boundary effects on the local critical behaviour of second-order phase transitions have become the subject of active study (Binder 1983), in part because of their relevance in wetting, adsorption, and phenomena in pores. It is well known by now that the presence of boundaries in a system near criticality often produces new singularities in local quantities. For example, the local susceptibility near a boundary has singular behaviour that depends on the boundary conditions (or surface interactions). In addition, some critical exponents may also depend on the shape of the boundary. For instance, the exponent for the susceptibility near a wedge formed by two intersecting free surfaces varies with the opening angle (Cardy 1983). For these local quantities, the concept of universality is a weaker notion than it is for bulk properties; it has to be defined for a given geometric shape and boundary condition.

The wedge exponents for the O($n$) spin model with Dirichlet boundary condition has been calculated to order $\varepsilon$ $(\varepsilon = 4-d)$ by using the field-theoretic renormalisation group method (Cardy 1983). The two- and three-dimensional edge critical behaviour of self-avoiding-walks (SAWs) and the Ising model have been investigated using series analysis (Guttmann and Torrie 1984). Real-space renormalisation group techniques have also been employed to study the Ising system with various boundary conditions

‡ Present address: Department of Chemistry, University of California, Los Angeles, CA 90024, USA.

0305-4470/90/122575+11$03.50  

(Larsson 1986) and percolation near an edge (Saxena 1987). In addition, exact results have been offered for edge exponents in two dimensions for the Ising model ($n=1$), Potts model (Cardy 1984) and the SAW problem ($n=0$) (Cardy and Redner 1984, Duplantier and Saleur 1986) with Dirichlet boundary condition by using conformal symmetries (Dotsenko 1984, Cardy 1987). These conformal methods, however, have not proved useful in studying higher-dimensional systems (the most relevant being $d=3$), where field-theoretic renormalisation methods are still the most powerful and systematic tools in understanding critical phenomena.

This paper studies the local critical behaviour of an O($n$) spin model bounded by $p$ mutually perpendicular surfaces. The semi-infinite case corresponds to $p=1$, the wedge geometry to $p=2$ and the corner geometry to $p=3$. The model is described by the standard Ginzburg-Landau free energy functional

$$F[\phi(r)]=\int_v \mathrm{d}^d r[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}t\phi^2+(1/4!)g_0(\phi^2)^2] \tag{1}$$

together with geometric constraints and appropriate boundary conditions. We consider two types of boundary conditions: von Neumann and Dirichlet. These boundary conditions can be derived from (1) by adding a surface term $\int_s \mathrm{d}^{d-1}r\frac{1}{2}c\phi^2$, with the former corresponding to $c=0$ and the latter to $c=\infty$. (Intermediate values of $c$ are driven to the fixed point $c=\infty$ under the renormalisation group transformation (Burkhardt and Eisenriegler 1981, Diehl 1986).) Since we use dimensional regularisation with minimal subtraction, the mean field $c=0$ value for the special transition persists to all orders of perturbatuon theory (Diehl 1986). Other regularisation schemes, such as cutoff regularisation, produce a shift in $c$ from its mean field value. The semi-infinite model ($p=1$) has been studied extensively (Diehl and Dietrich 1980, 1981, 1983, Diehl 1986); $p=2$ with Dirichlet boundary conditions is a special case of that studied by Cardy (1983); the Ising model ($n=1$) with von Neumann boundary condition has been partially studied by Larsson (1986); whereas the other five possibilities (allowing mixed boundary surfaces) have not been previously considered.

In the following sections, we study the singular behaviour of the local field, the correlation functions, and various response functions near the intersection of various surfaces. We calculate the local susceptibility exponent $\gamma$ to order $\varepsilon$ using renormalisation group methods. Then, combining the value of $\gamma$ with the scaling relations derived in the next section, we obtain the anomalous scaling dimension for the local field. The latter enables us to derive the scaling behaviour of any two-point spin-spin correlation functions and response functions.

To facilitate the calculation, the Green function $G^{(0)}(r, r'; t)$ for the free field ($g_0=0$) is written as the Laplace transform of the Green function $G^{(0)}(r, r'; N)$ for an ideal (Gaussian) polymer chain, with $t$ the Laplace conjugate variable to the chain length $N$ (Eisenriegler *et al* 1982, Freed 1987). The polymer representation is convenient because $G^{(0)}(r, r'; N)$ factors into its cartesian components

$$G^{(0)}(r, r'; N)=\prod_{i=1}^{d} G^{(0)}(x_i, x_i'; N) \tag{2}$$

where $d$ is the dimensionality of the system. This last feature enables us to express the zeroth-order Green function for our geometry as a product of $d$ simple one-dimensional Green functions, each satisfying its own boundary condition. The propagator $G^{(0)}(r, r'; t)$, on the other hand, is non-separable and quite unwieldy for computations. Use of the representation (2) also facilitates consideration of intermediate values of $c$ for $0 < c < \infty$. The polymer-magnet correspondence has been employed in the past to study the properties of polymers by using results derived in the magnet context. Here we use it conversely to study the scaling behaviour of the magnet (spin) system by calculating the corresponding polymer problem which is mathematically much simpler to treat for the particular geometries in consideration.

## 2. Scaling analysis

Consider the two-point correlation function between the field $\phi(r)$ at $r$ and the field $\phi(r')$ at $r'$:

$$G(r, r'; t) = \langle \phi(r)\phi(r') \rangle. \tag{3}$$

Since the canonical (mean field) scaling dimension of the field is $L^{1-d/2}$, where $L$ denotes the dimension of length, the correlation function (3) has the dimension $L^{2-d}$. If one of the fields, say $\phi(r)$, is located at a $c = \infty$ surface, then the correlation function vanishes because of the Dirichlet boundary condition. In order to obtain a meaningful correlation function in this case, it is necessary either to replace $\phi$ by $(1/c)(\partial\phi/\partial z)$, where $z$ is the coordinate normal to the surface, keeping $c$ large but finite, or, alternatively, to evaluate the correlation function at a distance $a$ away ($a \to 0$) from the surface by replacing $\phi$ by $a(\partial\phi/\partial z)$ (Diehl and Dietrich 1981, Diehl 1986). These two approaches are completely equivalent, apart from an unimportant prefactor. We adopt the second choice here. Because the normal derivative reduces the dimension by unity, the field near a Dirichlet surface has the reduced scaling dimension $L^{-d/2}$. More geneally, it can be shown (see the next section) that, if the field $\phi$ is near the intersection of $n_D$ orthogonal Dirichlet surfaces, it should be replaced by $a_1 \ldots a_{n_D}(\partial^{n_D}\phi/\partial z_1 \ldots \partial z_{n_D})$, and hence the canonical dimension reduces by $n_D$. Taking these cases into account, the correlation function has the scaling behaviour

$$G(r, r'; t) \sim L^{2-d-n_D-n'_D}. \tag{4}$$

Near criticality, fluctuations produce anomalous scaling dimension to the field. Then (4) should be modified to read

$$G(r, r'; t) \sim L^{2-d-n_D-n'_D+h+h'}, \tag{5}$$

where $h$ and $h'$ are, respectively, the anomalous scaling dimension of the two fields at $r$ and $r'$, respectively. (We have assumed that $r$ and $r'$ are far apart. If $r' \to r$, then the field operators coalesce to the composite operator $\phi^2(r)$ which in general has a different anomalous scaling dimension (Amit 1984).)

Linear response theory implies that the two-point correlation function in (3) is also the response function of the field at $r$ to an infinitesimal external field at $r'$. Such response (susceptibility) functions to the applied surface field, bulk field, etc, may be obtained by integrating (3) over appropriate coordinates in the codimension $d'$. Then these susceptibility functions have the following scaling behaviour:

$$\chi \sim L^{2-d-n_D-n'_D+h+h'+d'}. \tag{6}$$

The susceptibility exponent $\gamma$ is defined as

$$\chi \sim t^{-\gamma} \tag{7}$$

where $t$ is the usual reduced temperature. Since $t$ is related to the length $L$ by $L \sim t^{-\nu}$, with $\nu$ being the usual correlation length exponent, (6) is written in $t$ as

$$\chi \sim t^{-\nu(2-d-n_D-n'_D+h+h'+d')}. \tag{8}$$

Comparing (8) with (7) one identifies the susceptibility exponent $\gamma$ as

$$\gamma = \nu(2-d-n_D-n'_D+h+h'+d'). \tag{9}$$

The counterparts of these susceptibility functions for the polymers are the various restricted partition functions $T$ with one or both ends fixed at the boundary surface(s) (Eisenriegler *et al* 1982, Nemirovsky and Freed 1985, Wang *et al* 1987). For example, the susceptibility function of a field at the edge to the bulk external field is related to the partition function of a polymer chain with one end fixed at the edge and the other end free. Mathematically, these partition functions for the polymer of length $N$ are obtained by inversely Laplace transforming the corresponding susceptibility functions in the $n \to 0$ limit of the O($n$) spin model:

$$T(N) = \lim_{n\to 0} \mathscr{L}^{-1}\{\chi(t)\} \tag{10}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform operator.

In the next section, we use the direct chain conformation RG method (Freed 1987) to calculate the polymer partition functions $T(N)$ with one end fixed at the corner, edge, surface, etc, while the other end is free. Thus, we have $d' = d$ and $h' = h_{\mathrm{bulk}}$, the latter being zero to order $\varepsilon$ (Amit 1984). However, we do not impose the $n \to 0$ limit. Then (9) enables us to find the anomalous dimension $h$ for the field at the corner, edge, etc, which in turn can be used to compute the exponent $\gamma$ when the other end is on the surface, near the edge, etc.

## 3. Order $\varepsilon$ RG calculation of the exponent $\gamma$

The perturbation expansion for the two-point function follows the standard Feynman rules (Amit 1984) which have been shown to remain unaltered by the presence of boundaries (Diehl 1986). To first order in the coupling parameter $g_0$, we have

$$G(r, r'; t) = G^{(0)}(r, r'; t) - \tfrac{1}{6}(n+2)g_0 \int \mathrm{d}r_1\, G^{(0)}(r, r_1; t)G^{(0)}(r_1, r_1; t)G^{(0)}(r_1, r'; t) + \mathrm{O}(g_0^2) \tag{11}$$

where $G^{(0)}$ is the zeroth-order (unperturbed) two-point function that satisfies the appropriate boundary conditions and $n$ is the number of components of the field. After an inverse Laplace transform (11) becomes

$$G(r, r'; N) = G^{(0)}(r, r'; N) - \tfrac{1}{6}(n+2)g_0 \int_0^N \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \int \mathrm{d}r_1 G^{(0)}(r, r_1; N-\tau_2) \times G^{(0)}(r_1, r_1; \tau_2-\tau_1)G^{(0)}(r_1, r'; \tau_1) + \mathrm{O}(g_0^2). \tag{12}$$

By using the property (2), $G^{(0)}(r, r'; N)$ can be written explicitly as

$$G^{(0)}(r, r'; N) = (2\pi^{1/2}N^{1/2})^{-d} \prod_{i=1}^{d-n_N-n_D} \exp[-(x_i - x_i')^2/4N]$$
$$\times \prod_{j=1}^{n_N} \{\exp[-(y_j - y_j')^2/4N] + \exp[-(y_j + y_j')^2/4N]\}$$
$$\times \prod_{k=1}^{n_D} \{\exp[-(z_k - z_k')^2/4N] - \exp[-(z_k + z_k')^2/4N]\} \tag{13}$$

where $\{y_j\}$ denores the coordinates normal to the von Neumann surfaces, $\{z_k\}$ denotes the coordinates normal to the Dirichlet surfaces, and $\{x_i\}$ denotes the translationally invariant coordinates parallel to the surfaces. Although the polymer analogue of (13) corresponds to the $n \to 0$ limit of the free O($n$) field, the relations (11)-(13) are quite general for the O($n$), $n \neq 0$, field theory as $n$ enters only as a coefficient of the one-loop contribution in (11) and (12).

Equation (13) is simply the Green function for an ideal polymer chain with contour length $2N$. The universal part of partition function for the chain with an end fixed at $r'$ is simply

$$T(N) = \int \mathrm{d}r\, G(r, r'; N). \tag{14}$$

We set $x_i' = 0$, $y_j' = 0$ and $z_k' = a$ $(a \to 0)$. The integration of the zeroth-order term in (12) can be easily performed to yield

$$T^{(0)}(N) = [a/\pi^{1/2}N^{1/2}]^{n_D} \tag{15}$$

where we have used the relation

$$\exp[-(z_k - z_k')^2/4N] - \exp[-(z_k + z_k')^2/4N]$$
$$= (az/N) \exp[-z^2/4N] \qquad (z' = a \to 0). \tag{16}$$

In the first-order perturbation term, the translationally invariant coordinates are easily integrated first. Then, we use the fact that the integrations over the $y_j$ and $z_k$ are all uncoupled from each other, to arrive at

$$T^{(1)} = \int_0^N \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 [2\pi^{1/2}(N - \tau_2)^{1/2}]^{-(n_N+n_D)}[2\pi^{1/2}(\tau_2 - \tau_1)^{1/2}]^{-d}$$
$$\times (2\pi^{1/2}\tau_1^{1/2})^{-(n_N+n_D)} H_N^{n_N} H_D^{n_D} \tag{17}$$

where the function $H_N$ and $H_D$ are, respectively,

$$H_N = 2\int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}z' \{\exp[-(z - z')^2/4(N - \tau_2)] + \exp[-(z + z')^2/4(N - \tau_2)]\}$$
$$\times \{1 + \exp[-z'^2/(\tau_2 - \tau_1)]\} \exp[-z'^2/(4\tau_1)]$$
$$= 2\pi^{1/2}(N - \tau_2)^{1/2} 2\pi^{1/2}\tau_1^{1/2}[1 + (\tau_2 - \tau_1)^{1/2}/(3\tau_1 + \tau_2)^{1/2}] \tag{18a}$$

$$H_D = \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}z' \{\exp[-(z - z')^2/4(N - \tau_2)] - \exp[-(z + z')^2/4(N - \tau_2)]\}$$
$$\times \{1 - \exp[-z'^2/(\tau_2 - \tau_1)]\}(az'/\tau_1) \exp[-z'^2/(4\tau_1)]$$
$$= 2\pi^{1/2}(N - \tau_2)^{1/2} 2\pi^{1/2}\tau_1^{1/2} a\pi^{-1/2}\{[N - (\tau_2 - \tau_1)]^{-1/2}$$
$$- (\tau_2 - \tau_1)^{3/2}(3\tau_1 + \tau_2)^{-1}[4\tau_1(N - \tau_2) + (\tau_2 - \tau_1)(N - \tau_2 + \tau_1)]^{-1/2}\} \tag{18b}$$

and $T^{(1)}$ is defined as the first-order contribution to (12) divided by $-(n+2)g_0/6$. Then performing integration over $z$ and $z'$, (17) becomes

$$T^{(1)}=\int_0^N \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 [2\pi^{1/2}(\tau_2-\tau_1)^{1/2}]^{-d} F_N^{n_N} F_D^{n_D} \tag{19}$$

with the integrand factors

$$F_N = 1+(\tau_2-\tau_1)^{1/2}/(3\tau_1+\tau_2)^{1/2} \tag{20a}$$

$$F_D = a\pi^{-1/2}\{[N-(\tau_2-\tau_1)]^{-1/2}-(\tau_2-\tau_1)^{3/2}(3\tau_1+\tau_2)^{-1}[4\tau_1(N-\tau_2) + (\tau_2-\tau_1)(N-\tau_2+\tau_1)]^{-1/2}\}. \tag{20b}$$

Equation (19) can be brought into a more convenient form by defining the dimensionless variables $t=\tau_2/N$, $s=1-\tau_1/\tau_2$, and by writing $\varepsilon=4-d$. This converts (19) into

$$T^{(1)}=(2\pi^{1/2})^{-d}N^{\varepsilon/2}\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s\, t^{-1+\varepsilon/2} s^{-2+\varepsilon/2} F_N^{n_D} F_D^{n_N} \tag{21}$$

with

$$F_N = 1+\tfrac{1}{2}s^{1/2}(1-3s/4)^{-1/2} \tag{22a}$$

$$F_D = a\pi^{-1/2}N^{-1/2}\{(1-st)^{-1/2}-\tfrac{1}{8}s^{3/2}(1-3s/4)^{-1}[1-3s/4-t(1-s/2)^2]^{-1/2}\}. \tag{22b}$$

The integral in (21) is evaluated by following the procedure of dimensional regularisation ('t Hooft and Veltman 1972, Amit 1984, Freed 1987). To obtain the exponent $\gamma$ to order $\varepsilon$, it is only necessary to extract the residue of the $\varepsilon^{-1}$ pole. Clearly, the divergences arise from the $s\to 0$, $t\to 0$ behaviour of the integrand in (21). It is easy to see that the $\varepsilon^{-1}$ terms come from the forms $s^1 g(t)$ and $t^0 f(s)$ in $F_N^{n_N} F_D^{n_D}$, where $f(s)$ and $g(t)$ are regular functions in the limits $s\to 0$ and $t\to 0$, respectively. However, when $g(t)=$ constant + higher powers of $t$, a double pole arises, since the $s$ and $t$ integrals both have $\varepsilon^{-1}$ poles. This happens when $n_N\geqslant 2$, as demonstrated in the following.

For simplicity, let us first consider the case $n_D=0$. For $n_N\geqslant 2$, (22$a$) yields

$$F_N^{n_N} = 1+\tfrac{1}{2}n_N s^{1/2}(1-3s/4)^{-1/2}+\tfrac{1}{2}n_N(n_N-1)\tfrac{1}{4}s(1-3s/4)^{-1}+\ldots. \tag{23}$$

The $t$-integration in (21) can be easily performed to yield

$$\int_0^1 \mathrm{d}t\, t^{-1+\varepsilon/2}=2\varepsilon^{-1}. \tag{24}$$

In the $s$-integration, it can be seen that all other terms in (23) except the third term yield a finite contribution, while the third term produces

$$\begin{aligned}\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2}\tfrac{1}{2}n_N(n_N-1)\tfrac{1}{4}s(1-3s/4)^{-1} &= \tfrac{1}{8}n_N(n_N-1)\int_0^1 \mathrm{d}s\, s^{-1+\varepsilon/2}+\text{finite terms}\\ &= \tfrac{1}{8}n_N(n_N-1)2\varepsilon^{-1}+\text{finite terms}.\end{aligned} \tag{25}$$

Thus we see the emergence of the double pole $\frac{1}{8}n_N(n_N-1)\ (2\varepsilon^{-1})^2$ for $n_N \geqslant 2$. This pathological behaviour is not remedied by $F_D^{n_D}$ as the first term in the expansion of $F_D^{n_D}$ in ascending powers of $s$ and $t$ is a constant independent of $\varepsilon$. Calculations of the moments of the two-point correlation function $G(r, r'; t)$ or, equivalently, of the polymer analogue end-vector distribution shows that these moments are free of the double pole term and any associated additional order $\varepsilon^{-1}$ singularities (Myers *et al* 1990). Thus it appears that the double-pole term represents an additive contribution to the free energy. However, the (physical) origin of this double pole remains to be understood. The results of our calculations in this section are valid for $n_N < 2$ but should be taken as tentative for $n_N \geqslant 2$.

For $n_N < 2$ ($n_N = 0$ or 1), the divergent part of (21) can be easily extracted. In this case the expansion in (23) terminates after the second term, whereas for $F_D$ we write

$$F_D = a\pi^{-1/2}N^{-1/2}[1+\tfrac{1}{2}st+O(s^2t^2)-\tfrac{1}{8}s^{3/2}(1-3s/4)^{-3/2}+O(s^{3/2}t)]. \tag{26}$$

Thus we obtain

$$\begin{aligned}F_N^{n_N}F_D^{n_D} &= (a\pi^{-1/2}N^{-1/2})^{n_D}[1+\tfrac{1}{2}n_N s^{1/2}(1-3s/4)^{-1/2}] \\ &\quad\times[1+\tfrac{1}{2}st-\tfrac{1}{8}s^{3/2}(1-3s/4)^{-3/2}+O(s^{3/2}t)+O(s^2t^2)]^{n_D}\end{aligned} \tag{27}$$

and

$$\begin{aligned}T^{(1)} &= (2\pi^{1/2})^{-d}N^{\varepsilon/2}(a\pi^{-1/2}N^{-1/2})^{n_D}\int_0^1 \mathrm{d}t\, t^{-1+\varepsilon/2}\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2} \\ &\quad\times\Bigg[1+\tfrac{1}{2}n_D st+\tfrac{1}{2}n_N s^{1/2}(1-3s/4)^{-1/2} \\ &\quad+\sum_{i=1}^{n_D}\frac{n_D!}{i!(n_D-i)!}(-\tfrac{1}{8})^i s^{3i/2}(1-3s/4)^{-3i/2}+\tfrac{1}{2}n_N\sum_{i=1}^{n_D}\frac{n_D!}{i!(n_D-i)!} \\ &\quad\times(-\tfrac{1}{8})^i s^{(3i+1)/2}(1-3s/4)^{-(3i+1)/2}\Bigg]+\text{convergent terms}\end{aligned} \tag{28}$$

upon expanding the second bracket in (27).

The first and second terms can be integrated to yield

$$\begin{aligned}\int_0^1 \mathrm{d}t\, t^{-1+\varepsilon/2}\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2}[1+(1/2)n_D st] &= 2\varepsilon^{-1}/(-1+\varepsilon/2) \\ &\quad+(n_D/2)2\varepsilon^{-1}/(1+\varepsilon/2)\end{aligned} \tag{29}$$

while the remaining $s$-integrals all have the form

$$\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2}s^{\lambda}(1-3s/4)^{-\lambda}. \tag{30}$$

For $\lambda > 1$, the integral is evaluated by expanding the power $s^{\varepsilon/2}$ in $\varepsilon$ as

$$\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2}s^{\lambda}(1-3s/4)^{-\lambda} = \int_0^1 \mathrm{d}s\, s^{-2}s^{\lambda}(1-3s/4)^{-\lambda}+O(\varepsilon). \tag{31}$$

Changing variable $v = s^{-1}$, the first term in (31) becomes

$$\int_1^{\infty} \mathrm{d}v(v-\tfrac{3}{4})^{-\lambda} = (\lambda-1)^{-1}4^{\lambda-1}. \tag{32}$$

For $\lambda < 1$, we integrate (31) by parts to write

$$\int_0^1 \mathrm{d}s\, s^{-2+\varepsilon/2} s^{\lambda}(1-3s/4)^{-\lambda}$$
$$= (-1+\varepsilon/2+\lambda)^{-1}(s^{-1+\varepsilon/2+\lambda}(1-3s/4)^{-\lambda}|_0^1$$
$$-(3\lambda/4)\int_0^1 \mathrm{d}s\, s^{-1+\varepsilon/2+\lambda}(1-3s/4)^{-\lambda-1}). \tag{33}$$

The first term on the right-hand side of (31) is evaluated for $\varepsilon > 2$ and then analytically continued to $\varepsilon > 0$. The last integral in (33) can now be calculated by using (31) and (32). It is thus found that the order $\varepsilon^0$ term still has the form of (32). Hence, (32) applies for both $\lambda < 1$ and $\lambda > 1$, but becomes divergent when $\lambda \to 1$. The latter is exactly what happens when $n_N \geq 2$.

Combining all these results yields $T^{(1)}$ as

$$T^{(1)} = (2\pi^{1/2})^{-d} N^{\varepsilon/2}(a\pi^{-1/2}N^{-1/2})^{n_D}[-2\varepsilon^{-1} + n_D\varepsilon^{-1} - \varepsilon^{-1}(n_N + I_1(n_D) + n_N I_2(n_D))]$$
$$+ O(\varepsilon^0) \tag{34}$$

and the partition function $T(N)$ of (14) becomes

$$T = T^{(0)} - \tfrac{1}{6}(n+2)g_0 T^{(1)}$$
$$= (a\pi^{-1/2}N^{-1/2})^{n_D}[1 + \tfrac{1}{6}(n+2)g_0(2\pi^{1/2})^{-d}N^{\varepsilon/2}2\varepsilon^{-1}$$
$$\times (1 + n_N/2 + I_1(n_D)/2 - n_D/2 + n_N I_2(n_D)/2)] + O(\varepsilon) \tag{35}$$

where the functions $I_1(n_D)$ and $I_2(n_D)$ are defined as

$$I_1(n_D) = \sum_{i=1}^{n_D} \frac{n_D!}{i!(n_D-i)!}\frac{(-1)^{i+1}}{3i-2} = \frac{1}{2}\left(\frac{\Gamma(\frac{1}{3})\Gamma(n_D+1)}{\Gamma(n_D+\frac{1}{3})} - 1\right) \tag{36}$$

$$I_2(n_D) = \sum_{i=1}^{n_D} \frac{n_D!}{i!(n_D-i)!}\frac{(-1)^{i+1}}{3i-1} = \frac{\Gamma(\frac{2}{3})\Gamma(n_D+1)}{\Gamma(n_D+\frac{2}{3})} - 1. \tag{37}$$

For $n_D = 1, 2, 3$, $I_1(n_D) = 1, \frac{7}{4}, \frac{67}{28}$ and $I_2(n_D) = \frac{1}{2}, \frac{4}{5}, \frac{41}{40}$, respectively.

We define the dimensionless coupling parameter

$$u_0 = g_0(2\pi)^{-d} S_d \kappa^{-\varepsilon} \tag{38}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a $d$-dimensional hypersphere of unit radius and $\kappa^{-1}$ is some phenomenological length scale. Removing the divergence following the standard renormalisation procedure (Amit 1984, Freed 1987), expanding $(\kappa^2 N)^{\varepsilon/2}$ in powers of $\varepsilon$ and re-exponentiating at the fixed point $u^* = 6\varepsilon/(n+8)$ (Amit 1984), we obtain

$$T = (a\pi^{-1/2})^{n_D} N^{-n_D/2}(\kappa^2 N)[(n+2)/2(n+8)]\varepsilon(1 + n_N/2 + I_1(n_D)/2 - n_D/2$$
$$+ n_N I_2(n_D)/2) + O(\varepsilon) \tag{39}$$

from which we identify the exponent $\gamma$ as

$$\gamma = 1 - n_D/2 + \frac{n+2}{2(n+8)}\varepsilon(1 + n_N/2 + I_1(n_D)/2 - n_D/2 + n_N I_2(n_D)/2) + O(\varepsilon^2). \tag{40}$$

The result (40) reproduces the known values of the exponent $\gamma$ for $n_N = n_D = 0$ (bulk), $n_N = 1$, $n_D = 0$ (semi-infinite, special), and $n_N = 0$, $n_D = 1$ (semi-infinite, ordinary). It is interesting to observe the appearance of a coupled term in (40) when $n_N \neq 0$, $n_D \neq 0$. For instance, a 90° wedge formed by one van Neumann surface and one Dirichlet surface yields

$$\gamma = \frac{1}{2} + \frac{7}{8}\frac{(n+2)}{(n+8)}\,\varepsilon + O(\varepsilon^2). \tag{41}$$

The anomalous dimension for the field near the intersection of $n_N$ von Neumann surfaces and $n_D$ Dirichlet surfaces is obtained from (9) by setting $d' = d$, in which case $h' = 0 + O(\varepsilon^2)$ and

$$h = \gamma/\nu + n_D - 2 = \frac{n+2}{2(n+8)}\,\varepsilon(n_N + I_1(n_D) + n_N I_2(n_D)) + O(\varepsilon^2). \tag{42}$$

This anomalous dimension (42) enables us to obtain the exponent $\gamma$ for the case where the other end is not free, but is constrained to lie on the intersection of $(n'_N + n'_D)$ surfaces. Then $d' = d - (n'_N + n'_D)$, and using (9), we find

$$\gamma = 1 - \tfrac{1}{2}(d - d' + n_D + n'_D) + \frac{n+2}{2(n+8)}$$
$$\times \varepsilon[1 + (d'-d)/2 + n_N/2 + I_1(n_D)/2 - n_D/2 + n_N I_2(n_D)/2 + n'_N/2$$
$$+ I_1(n'_D)/2 - n'_D/2 + n'_N I_2(n'_D)/2]. \tag{43}$$

It is easily verified that (43) reproduces the familiar $\gamma$ values for the bulk $\gamma_b$ ($n_N = n_D = n'_N = n'_D = 0$), ordinary $\gamma_{1,1}^{\mathrm{ord}}$ ($n_N = n'_N = 0$, $n_D = n'_D = 1$), and special $\gamma_{1,1}^{\mathrm{sp}}$ ($n_N = n'_N = 1$, $n_D = n'_D = 0$). For the interesting case where both ends are on the edge formed by a von Neumann surface and a Dirichlet surface ($n_N = n_D = n'_N = n'_D = 1$), we have the new result

$$\gamma = -1 + \frac{3}{4}\frac{(n+2)}{(n+8)}\,\varepsilon + O(\varepsilon^2). \tag{44}$$

In the case of two intersecting Dirichlet surfaces, our results disagree with those of Cardy (1983) for $\alpha = \pi/4$. Finally, we emphasise that the above calculations for $n_N \geq 2$ should be taken as tentative because of our need to properly treat the double-pole contributions.

## 4. Concluding remarks

We have calculated a class of critical exponents for the local susceptibilities of the O($n$) spin model, which arises from the presence of a surface, an edge, a corner, or combinations thereof. Our results illustrate the importance of these geometric conditions, as well as the role of surface interactions, in governing the local critical behaviour of the system. A novel feature of our calculation is that we consider a variety of mixed von Neumann and Dirichlet boundary conditions. For example, a wedge may be formed by a von Neumann surface and a Dirichlet surface. In such cases, the singular behaviour (anomalous scaling dimension) of the local field contains contributions from each boundary condition as well as a coupling term which reflects the combined effects of the von Neumann and Dirichlet boundary conditions (cf (40)). When the intersection consists of two or more von Neumann surfaces, a pathological $\varepsilon^{-2}$ pole appears in the *first-order* $\varepsilon$ expansion. We have recently found that such double pole is not only present in the 90° geometry, but it persists to a wedge of any opening angle $\pi/n$ ($n = 2, 3, 4, \ldots$) with an angle-dependent residue (Myers *et al* 1990). This double pole can be removed by absorbing it to the zero point of the free energy (additive renormalisation), as is supported by the absence of such double poles and any associated additional order $\varepsilon^{-1}$ poles in the calculations of the *moments* of the two-point correlation function. Nevertheless, it remains necessary to understand the physical origin of the appearance of this double pole. It is possible that the theory with two or more intersecting von Neumann surfaces are not renormalisable in its present form and that additional edge and corner operators are required to produce a finite theory. A formal proof of the renormalisability of the theory, however, is clearly beyond the scope of the present paper.

We would like to comment that similar unexpected features have appeared in a number of studies of the edge critical behaviour. Cardy (1983) pointed out some difficulties with his $\varepsilon$ expansion at the edge ordinary transition that occur when the opening angle $\alpha < (5\pi/24)[(n+2)/(n+8)]\varepsilon + O(\varepsilon^2)$. Larsson, using real-space RG, finds that if the angle between the two ordinary (special) surfaces is less (larger) than 53° (297°), the edge fixed point disappears (diverges) (Larsson 1986). He interprets the second case as a signal of the occurence of a first-order transition. On the other hand, the calculations of Guttmann and Torrie (1984) and Saxena (1987) for the SAW and the Ising model, and for percolation, respectively, near the edge ordinary transition do not show similar difficulties. It is possible that some of these anomalies are interrelated and that they are some artefact of the $\varepsilon$ expansion for wedge geometries. On the other hand, they could also signal some new physical phenomena. Either way, further investigations are warranted.

## Acknowledgment

This research is supported in part by NSF grant DMR86-14358.

## References

Amit D J 1984 *Field Theory, the Renormalisation Group and Critical Phenomena* (Singapore: World Scientific)
Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8 ed C Domb and J L Lebowitz (New York: Academic) p 1
Burkhardt T W and Eisenriegler E 1981 *Phys. Rev.* B **24** 1236
Cardy J L 1983 *J. Phys. A: Math. Gen.* **16** 3617
—— 1984 *Nucl. Phys.* B **240** [FS12] 514
—— 1987 *Phase Transitions and Critical Phenomena* vol II, ed C Domb and J L Lebowitz (New York: Academic) p 55
Cardy J L and Redner S 1984 *J. Phys. A: Math. Gen.* **17** L933
Carslaw H S and Jaeger J C 1959 *Conduction of Heat in Solids* (Oxford: Clarendon)
Diehl H W 1986 *Phase Transitions and Critical Phenomena* vol 10, ed C Domb and J L Lebowitz (New York: Academic) p 75
Diehl H W and Dietrich S 1980 *Phys. Lett.* **80A** 408
—— 1981 *Z. Phys.* B **43** 315
—— 1983 *Z. Phys.* B **50** 117
Dotsenko V S 1984 *Nucl. Phys.* B **235** [FS11] 54
Duplantier B and Saleur H 1986 *Phys. Rev. Lett.* **57** 3179
Eisenriegler E, Kremer K and Binder K 1982 *J. Chem. Phys.* **77** 6296

Freed K F 1987 *Renormalisation Group Theory of Macromolecules* (New York: Wiley)
Gittmann A J and Torrie G M 1984 *J. Phys. A: Math. Gen.* **17** 3539
Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
Larsson T A 1986 *J. Phys. A: Math. Gen.* **19** 1691
Myers K R, Nemirovsky A M, Freed K F and Wang Z-G 1990 to be submitted.
Nemirovsky A M and Freed K F 1985 *J. Chem. Phys.* **83** 4166
Saxena V K 1987 *J. Phys. A: Math. Gen.* **20** 6623
't Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189
Wang Z-G, Nemirovsky A M and Freed K F 1987 *J. Chem. Phys.* **86** 4266